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GENERATING COVARIANCE SEQUENCES AND THE CALCULATION OF QUANTIZA--ETC(U)  
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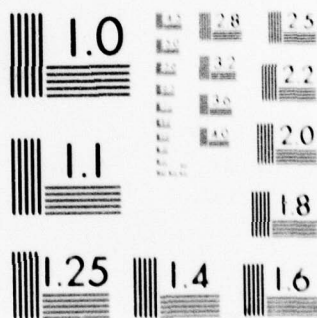
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Generating Covariance Sequences and the Calculation of  
Quantization and Rounding Error Variances in Digital Filters.

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Generating Covariance Sequences and the Calculation of  
Quantization and Rounding Error Variances in Digital Filters<sup>1</sup>

by

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### Abstract

A linear algorithm is given for the generation of covariance sequences for rational digital filters using numerator and denominator coefficients directly. There is no need to solve a Lyapunov equation or to solve for the residues of a spectrum, as in other methods. By appealing to certain results from the theory of inners we show the algorithm provides a unique solution provided only that the filter is stable.

Our results may be used to compute error variances due to product rounding and signal quantization, and to generate covariance strings  $\{r_k\}_0^K$  used in other studies involving second-order properties of digital filters.

$\langle r_{\text{sub } k} \rangle$  from  $\phi$  to  $k$



# I. Introduction

There are numerous problems in the study of digital filters that require solution of an integral like

$$r_0 = \frac{1}{2\pi j} \oint H(z)H(z^{-1})\frac{dz}{z} \quad (1)$$

or, more generally, like

$$r_k = \frac{1}{2\pi j} \oint H(z)H(z^{-1})z^k \frac{dz}{z} \quad (2)$$

For example, in the computation of steady-state output noise variance  $\sigma^2$  due to fixed-point quantization, one needs  $\sigma^2 = (\Delta^2/12)r_0$ , where  $\Delta$  is the quantization step size and  $H(z)$  is the transfer function between the quantization noise source and the filter output. The same kind of computation is required for the computation of output noise variance due to input signal quantization. More generally, sequences like  $\{r_k\}_0^K$  are required in the study of model reduction procedures for approximating high-order filters with low-order ones.

The most obvious way to compute  $r_k$  is to perform the indicated contour integration by evaluating residues. Mitra, et. al. [2] have tabulated the appropriate residues for generic terms arising in the computation of  $r_0$ . With some tedium the results may be generalized to handle the calculation of  $r_k$ . To apply the results of [2], one must solve for poles of  $H(z)$ . This can be a nuisance, so one looks for alternatives.

Note  $r_k$  may be written [1]

$$r_k = h_0 h_k + c' \phi^k K c, \quad k \geq 0 \quad (3)$$

where  $K$  is the solution to a matrix Lyapunov equation,  $\phi^k$  is a state transition matrix,  $\{h_k\}_0^\infty$  is the unit pulse response sequence for  $H(z)$ ,

and  $c' = (1 \ 0 \dots 0)$ . The computational difficulty with (3) is that a Lyapunov equation must be solved and  $\phi^k$  determined. The latter determination involves solving for the eigenvalues of  $\phi$ , performing  $N$  matrix multiplies to initialize the recursion to follow, or exploiting special properties of the matrix  $\phi$ . Another alternative is to proceed as Jury does [3] and write  $r_0$  as the ratio of two determinants. This alternative is not particularly attractive because it does not easily generalize to the computation of  $r_k$ .

Our approach follows.

## II. An Alternative Method

Let  $H(z)$  denote a stable autoregressive moving average digital filter of the form

$$H(z) = \frac{\sum_{m=0}^M b_m z^{-m}}{\sum_{\ell=0}^N a_\ell z^{-\ell}} ; \quad a_0 = 1 \quad a_1, b_1 \text{ real numbers} \quad (4)$$

$$\sum_{\ell=0}^N a_\ell z^{-\ell} = 0 \rightarrow |z| < 1$$

Denote this filter  $H(z)$  : ARMA( $N, M$ ). There is no need for  $M$  to be less than  $N$ . The covariance sequence associated with  $H(z)$  is  $\{r_k\}_{-\infty}^{\infty}$  with  $r_k = r_{-k}$  given by

$$r_k = \frac{1}{2\pi j} \oint_C S(z) z^k \frac{dz}{z} , \quad \forall k \quad (5)$$

$$S(z) = H(z)H(z^{-1})$$

Here the contour  $C$  lies within the region of absolute convergence of  $H(z)$ . The contour may be chosen to be the unit circle in which case  $S(z = \exp(j2\pi f))$  is the spectrum (or magnitude-squared frequency response) corresponding to  $H(z)$ .

The unit pulse sequence corresponding to  $H(z)$  is  $\{h_k\}_{-\infty}^{\infty}$  with

$$h_k = \begin{cases} 0, & k < 0 \\ b_k - \sum_{n=1}^N a_n h_{k-n}, & k \geq 0 \end{cases} \quad (6)$$

Here  $b_k$  is assumed zero for  $k \geq M+1$ . The covariance sequence is related to the unit pulse sequence as follows:

$$r_k = \sum_{n=0}^{\infty} h_n h_{n+k}, \quad k \geq 0 \quad (7)$$

$$r_k = r_{-k}$$

Substitute (6) into (7) to get

$$\sum_{n=0}^N a_n r_{k-n} = d_k, \quad k \geq 0 \quad (8)$$

$$d_k = \sum_{n=0}^{M-k} h_n b_{n+k}$$

Note  $d_k = 0$  for  $k \geq M+1$ , in which case the  $\{r_k\}$  sequence behaves just like a purely autoregressive one. That is, the sequence  $\{r_k\}$  satisfies a linear homogeneous difference equation for  $k \geq M+1$ .

From (8) it is clear that the covariance sequence may be generated recursively. The trick is to initialize the recursion by finding  $r_k$  for  $0 \leq k \leq N$ . Write out (8) for  $k = 0, 1, \dots, N$ :

$$A\bar{r} = \bar{d}$$

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & & & a_N \\ a_1 & a_2+a_0 & a_3 & a_4 & & & a_N & 0 \\ a_2 & a_3+a_1 & a_4+a_0 & a_5 & \dots & & a_N & 0 & 0 \\ a_3 & a_4+a_2 & a_5+a_1 & a_6+a_0 & \dots & a_N & 0 & . & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_N & a_{N-1} & \dots & \dots & \dots & \dots & a_1 & a_0 \end{bmatrix} \quad (9)$$



$$a_0 = 1$$

$$\bar{r}' = (r_0, r_1, \dots, r_N) \quad \bar{d}' = (d_0, d_1, \dots, d_N)$$

The matrix A is generated as follows: begin with the first row  $(a_0, a_1, a_2, \dots, a_N)$ ; left-shift this row  $(n-1)$  times and add the  $m^{\text{th}}$  overflow to the  $(n, m+1)$  term to get the  $n^{\text{th}}$  row. Thus the  $(i, j)$  element of matrix A is

$$\{A\}_{i,j} = \begin{cases} a_{i-1} & \text{for } j = 1 \\ a_{i+j-2} + a_{i-j} & \text{for } j > 1 \end{cases} \quad (10)$$

where  $a_0, a_1, \dots, a_n$  are defined in (4) and  $a_i = 0$  for  $i < 0$  and  $i > N$ . For example, the fourth row is  $(a_3, a_4+a_2, a_5+a_1, a_6+a_0, \dots, a_N, 0, \dots, 0)$ . See Appendix I of [4]. The right-most triangular region of the matrix consists of zeros.

The Method. The solution method is to generate  $\{h_k\}_0^M$  from (6), solve (9) for  $\bar{r}$ , and then use  $\{h_k\}$ ,  $\{a_n\}_0^N$ , and  $\{b_m\}_0^M$  to generate an arbitrarily long finite-length version of  $\{r_k\}_{-\infty}^\infty$  from (8). If only  $\{r_k\}_0^N$  is required (as in applications requiring only  $r_0$ ), then only (9) must be solved using standard techniques for solving linear equations.

The same procedures were outlined in [4] for generating covariance sequences for autoregressive filters. For autoregressive filters,  $d_0 = b_0^2$  and  $d_k = 0$ ,  $k > 0$ . The matrix A has also arisen in Jury's work [5].

### III. Comments on Unicity

The connection between the  $\{a_n\}_0^N$  and  $\{b_m\}_0^M$  parameters of an ARMA(N,M) filter  $H(z)$  and the corresponding covariance sequence  $\{r_k\}_{-\infty}^\infty$  is unique, provided  $H(z)$  is stable and minimum phase and there are no pole-zero cancellations. If  $H(z)$  is stable and there exist no pole-zero cancellations,

tions, but  $H(z)$  is not minimum phase, then corresponding to  $H(z)$  is a unique  $\{r_k\}_{-\infty}^{\infty}$ , but not vice-versa. Thus given the feedback and feed-forward parameters  $\{a_n\}_0^N$  and  $\{b_m\}_0^M$  for a stable  $H(z)$  one may solve for a unique  $\bar{r}$  from (9) and recursively get the corresponding unique covariance sequence  $\{r_k\}_{-\infty}^{\infty}$ . The following theorem is arcane and relevant.

Theorem: The Matrix A is Nonsingular for Stable Filters. Let  $H(z)$  be the transfer function of a stable ARMA(N,M) filter, as defined in (4). That is, the roots of  $A(z) = \sum_{n=0}^N a_n z^{-n}$  lie strictly inside the unit circle  $|z| = 1$ . Then the matrix A of (9) is nonsingular and the solution for  $\bar{r}$  in (9) is unique.

Proof: Rotate the matrix A by  $\pi/2$  and interchange the order of columns to obtain the matrix

$$\Omega = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \dots & a_N \\ a_1 & a_2 + a_0 & a_3 + a_1 & a_4 + a_2 & \dots & a_{N-1} \\ a_2 & a_3 & a_4 + a_0 & a_5 + a_1 & \dots & \\ \vdots & & & 0 & & \\ \vdots & & & & & \\ \vdots & & & & & \\ a_N & 0 & 0 & 0 & \dots & 0 & a_0 \end{bmatrix} \quad (11)$$

and preserve  $|\Omega| = |A|$ . Define

$$\begin{aligned} A_1(z) &= z^N A(z) \\ &= \sum_{n=0}^N a_n z^{N-n} \\ &= \sum_{n=0}^N \alpha_n z^n \end{aligned} \quad (12)$$

$$\alpha_n = a_{N-n}$$

The roots of  $A_1(z)$  lie inside  $|z| = 1$  by virtue of the fact that the roots of  $A(z)$  do. Jury [5] shows that necessary and sufficient con-

ditions for the roots of  $A_1(z)$  to lie inside  $|z| = 1$  are

$$1) \quad A_1(+1) > 0, \quad (-1)^n A_1(-1) > 0$$

$$11) \quad \Delta_{n-1}' = X_{n-1}' Y_{n-1}' \text{ positive innerwise}$$

In order to relate the determinant of  $\Omega$  to this stability criterion, define

$$X_{n+1} = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & \dots & a_0 \\ 0 & a_n & a_{n-1} & & a_1 \\ 0 & & a_n & & \cdot \\ & & & & \cdot \\ 0 & & & & a_n \end{bmatrix}; \quad Y_{n+1} = \begin{bmatrix} 0 & 0 & \dots & & a_0 \\ & & & a_0 & a_1 \\ & & & & \cdot \\ & & a_0 & & \cdot \\ & & & & a_n \end{bmatrix} \quad (13)$$

It follows from Jury [5] that

$$2|\Omega| = |X_{n+1} + Y_{n+1}| = |(-1)^n A_1(1) A_1(-1) \Delta_{n-1}^-| \quad (14)$$

If  $H(z)$  is stable, it follows that  $A_1(\pm 1) \neq 0$ ,  $|\Delta_{n-1}^-| \neq 0$ , and therefore that  $|\Omega| = |A| \neq 0$ . Q.E.D.

The following second-order example is illustrative.

Example: Assume  $H(z)$  is second order. Then

$$A = \begin{bmatrix} 1 & a_1 & a_2 \\ a_1 & a_2+1 & 0 \\ a_2 & a_1 & 1 \end{bmatrix} \quad (15)$$

This matrix is nonsingular provided  $a_2 \neq 1$  and  $a_1 \neq \pm(a_2+1)$ . These conditions are illustrated in Figure 1. Thus  $A$  is nonsingular on the two dimensional plane  $(a_1, a_2)$ , minus the boundary lines illustrated. Note the interior of these lines is the region of stability for a second-order filter. Thus for a stable second-order filter the matrix



A is nonsingular. The implication does not go the other way. All the  $(a_1, a_2)$  pairs outside the illustrated boundaries also give nonsingular A, but unstable  $H(z)$ . Solution of (9) for such pairs gives  $\{r_k\}$  sequences that are not covariance sequences.

#### IV. Remarks

The results presented here may be used to compute error variances in finite word-length digital filters and to generate covariance strings  $\{r_k\}_0^K$  for use in model reduction procedures and the like. The computations seem simpler than those of [6] and generalize the results of [2]-[4]. Software is available from the authors upon written request.



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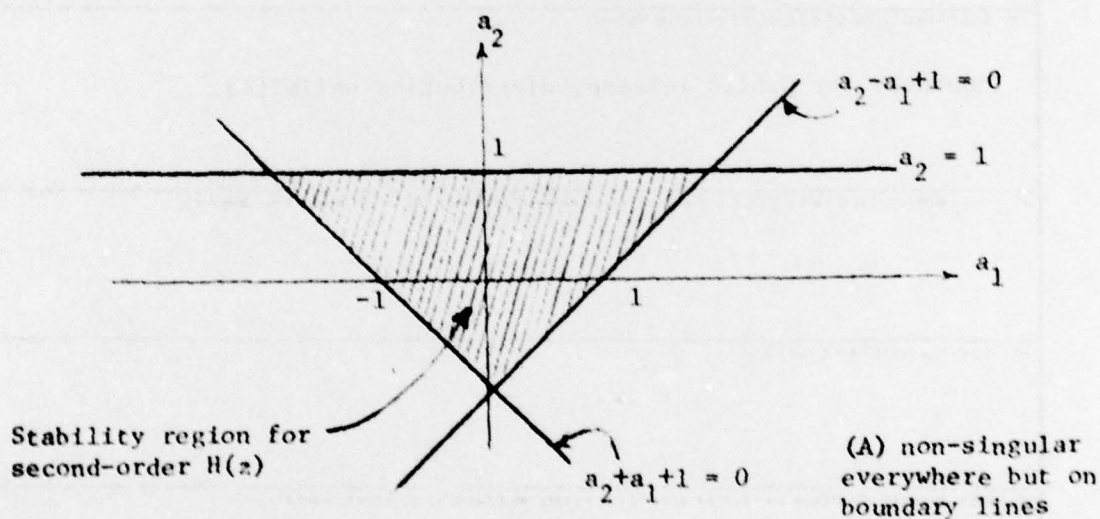


Figure 1. Regions of Stability and Nonsingularity

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20. rounding and signal quantization, and to generate covariance strings  $\{r_k\}_0^K$  used in other studies involving second-order properties of digital filters.

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